INTERACTION BETWEEN AN ACOUSTIC WAVE AND A DISC

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A numerical solution is obtained to the problem concerning a pressure measurement at the boundary between an ideal compressible fluid and a solid wall. It is assumed that the fluid occupies a semiinfinite cylinder with a rigid bottom into which an elastic disc is inserted and held firmly around its edges. Motion is produced by a pressure wave originating at infinity. A finite-difference grid for this application is described and the results of actual calculations are shown.

The elastic disc placed in the plane of the bottom wall and constrained around its edges serves as the sensing element in the measurement of stresses at the boundary between the continuous medium and that rigid wall. The stress in the medium is gaged by the deflection of this disc, assuming a linear relation between both quantities. The proportionality factor is assigned the value it has when the medium is an ideal fluid at rest. Obviously, such a method of measuring stress is fraught with inevitable errors which, in the best case, can only be minimized.

There are two basic sources of error here. First, under typical circumstances there may be shear stresses present in the medium. For example, the medium represents an elastic half-space bordering on a rigid stationary plane which contains a hole covered with the elastic disc. The medium is loaded and comes to rest. No matter how thin the disc is, its deflection under a fixed stress will obviously be limited. If the medium is a fluid, however, then under a fixed pressure the deflection will increase infinitely as the thickness of the disc approaches zero. Second, an error is produced if the stresses in the medium vary sufficiently fast. This error is simply due to the inertia of both the disc and the medium. These two errors are respectively defined as static and dynamic ones and, naturally, they may occur simultaneously, but it is more convenient to consider them separately.

The following analysis concerns the dynamic error only.

1. The simplest example of a medium free of any static error is an ideal compressible fluid. It is assumed here that such a medium occupies a semiinfinite circular cylinder with a rigid bottom into which a circular elastic disc is inserted held firmly around its edges (Fig. 1). Displacements in the medium are assumed small, and the equations describing them linear. The units of measure are chosen so that the radius of the disc, the density of the medium, and the acoustic velocity in the medium are all equal to one.

We use cylindrical coordinates z, r, θ, and time t. The medium occupies a semiinfinite cylinder $z > 0$, $r < R$, $R \geq 1$. The coordinates of the disc are $z = 0$, $r \leq 1$ (Fig. 1).

The motion within the medium is assumed a potential flow, with the potential ψ independent of θ and satisfying the wave equation

$$\psi_{tt} = \psi_{zz} + \Delta\psi \quad \text{for} \quad z > 0, \ r < R \quad (\Delta\psi = \psi_{rr} + \psi_r / r) \tag{1.1}$$

Displacements u_1 and u_2 and pressure p are related to ψ as follows: $u_1 = \psi_r$, $u_2 = \psi_z$, and $p = -\psi_{tt}$. The deflection of the disc is denoted by w. Continuity of displacements is maintained: $u_2 = w$ at $z = 0$, $t \geq 0$, and $r \leq 1$.

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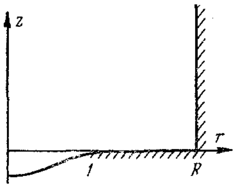

Fig. 1

The boundary conditions are defined as

$$\begin{gathered}\psi_r = 0 \quad \text{at} \quad r = R, \quad z > 0, \ \psi_z = 0 \quad \text{at} \quad z = 0, \ 1 < r < R \\ \psi_{ztt} + a_1 \Delta\Delta\psi_z - a_2\psi_{tt} = 0 \quad \text{at} \quad z = 0, \ r < 1 \\ \psi_z = 0, \ \psi_{zr} = 0 \quad \text{at} \quad z = 0, \ r = 1\end{gathered} \tag{1.2}$$

In dimensional variables

$$a_1 = \frac{El_1^2}{12(1-\sigma^2)\rho_2 l_2^2 c^2}, \quad a_2 = \frac{\rho_1 l_2}{\rho_2 l_1}$$

where l_1 is the radius, l_2 is the thickness, ρ_1 is the density, E is the Young modulus, and σ is the Poisson ratio of the disc, while ρ_2 and c are the density of the medium and the acoustic velocity in it.

The initial conditions at t = 0 are

$$\begin{gathered}\psi = -z^2/2, \ \psi_t = -z \quad \text{for} \quad z > 0, \ r < R \\ \psi_z = \psi_{zt} = 0 \quad \text{for} \quad z = 0, \ r < 1\end{gathered} \tag{1.3}$$

Such initial conditions correspond to a unit-step pressure originating at infinity.

This mathematical formulation of the problem is quite unwieldy, and a closed analytical solution has not been successfully obtained. Such problems usually reduce to integral equations which have to be solved by approximation or by numerical methods (see, e.g. [1]).

Here the problem will be solved by straight numerical computations. Since computations are feasible only for a finite and small number of parameter combinations, the problem requires a preliminary analysis, which will be carried out in the subsequent sections 2 and 3.

Of primary interest in the problem is the deflection of the disc at its center, as a function of time, since it is this variable which is measured in a practical test and which is used for gaging the pressure in the medium.

2. We now consider the solution to Eqs. (1.1) and (1.2) in the form*

$$\psi = \varphi(z, r)\exp(i\omega t), \quad \varphi \to 0 \quad \text{for} \quad z \to \infty, \ \varphi \geqslant 0 \tag{2.1}$$

The existence of a solution of the (2.1) kind signifies that the disc can vibrate without damping even if the pressure at infinity is zero and, consequently, the pressure at infinity may not be gaged by the displacement of the disc.

It will be shown further that system (1.1) and (1.2) cannot have a solution of the (2.1) kind. This fact seems quite obvious, but such solutions may exist with different boundary conditions.

We introduce function v(z) defined as

$$v = \int_0^R \varphi(z, r)\, r dr \tag{2.2}$$

From (1.1) and condition $\psi_r = 0$ at r = R we obtain an equation for v

$$\omega^2 v + v_{zz} = 0 \tag{2.3}$$

Obviously, this equation has no nontrivial solutions which vanish at infinity and, therefore, $v \equiv 0$. Since $\varphi \geq 0$, it follows from (2.2) that v(r) = 0 only if φ(z, r) = 0 for every r < R. Thus, system (1.1) and (1.2) has no nontrivial solutions of the (2.1) kind.

Let us consider boundary conditions which differ from (1.2) in that $\psi = 0$ at r = R and $\psi_z = \Delta\psi_z = 0$ at z = 0, r = 1. We will find the particular solution letting R = 1.

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*The inequality $\varphi \geq 0$ restricts somewhat the content of the statement which will be proved subsequently. It is entirely probable, however, that this statement will remain valid also when the sign of φ(r, z) changes. This is confirmed by the numerical results which yield the steady-state solution for $t \to \infty$. In the special case where R = 1 (the disc takes up the entire bottom of the "tube") a simple solution is possible by the method of separating the variables, and it becomes clearly evident here how the choice of boundary conditions affects the occurrence of free vibrations. In our example φ(r, z) certainly does not retain the same sign [Editor's note].

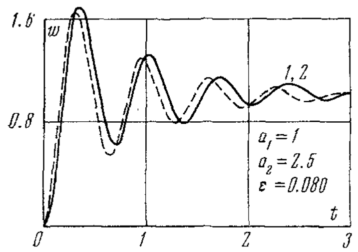

Fig. 2

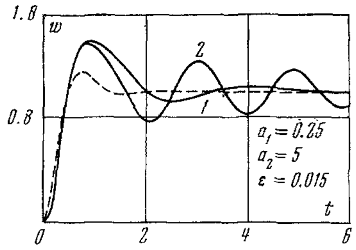

Fig. 3

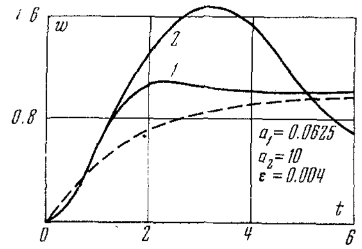

Fig. 4

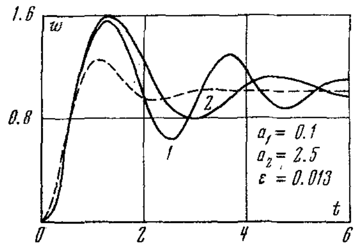

Fig. 5

The solution will be sought in the form

$$\psi = I_0 (\lambda r) \exp (i\omega t - \mu z) \tag{2.4}$$

where J_0 is a Bessel function, λ is the first root of equation $J_0(\lambda) = 0$, and μ and ω must be chosen so as to satisfy Eq. (1.1) and the boundary conditions.

If $\mu > 0$, then obviously (2.4) satisfies conditions (2.1).

Inserting (2.4) into (1.1) and (1.2), we obtain equations for ω and μ

$$\omega^2 + \mu^2 - \lambda^2 = 0, \qquad \mu (\omega^2 - a_1\lambda^4) + a_2\omega^2 = 0 \tag{2.5}$$

We will prove that (2.5) always has a solution. Indeed, considering condition $\mu > 0$, (2.5) can yield an equation for ω

$$\omega^2 - a_1\lambda^2 + a_2\omega^2 / \sqrt{\lambda^2 - \omega^2} = 0 \tag{2.6}$$

It is evident that the left-hand side of this equation is a monotonic function of ω when $0 < \omega < \lambda$, being negative when $\omega = 0$ but positive when $\omega \to \lambda$ and that, therefore (2.6) has a unique solution on the interval $(0, \lambda)$.

With μ and ω chosen so as to satisfy (2.5), then, (2.4) is a solution of the (2.1) kind.

3. We next obtain an approximate solution of the constraint problem (1.1), (1.2), and (1.3) with the additional assumption that R = 1. This solution describes the disc deflection as a function of time and contains no information about the pressure field.

We introduce function v(t, z) defined similarly as (2.2)

$$v(t, z) = 2 \int_0^1 \psi(t, z, r)\, r\, dr \tag{3.1}$$

Integrating Eq. (1.1) over r yields the following equation for v:

$$v_{tt} = v_{zz} \tag{3.2}$$

Essential use has been made here of the condition that $\partial\psi/\partial r = 0$ at $r = 0$, and $z > 0$.

Integrating (1.3), we obtain the initial conditions for (3.2)

$$v(0, z) = -z^2 / 2, \quad v_t(0, z) = -z \quad \text{for} \quad z > 0 \tag{3.3}$$

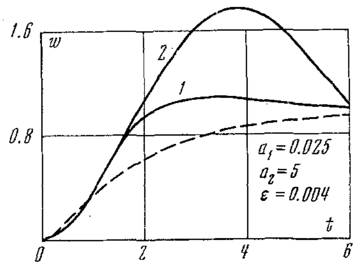

Fig. 6

For simplification, the disc deflection will be sought in the form

$$w = Q(t)\ (1 - r^2)^2 / 32 \tag{3.4}$$

where Q(t) is an unknown function. The relation between w and r has been defined thus, in order that (3.4) should correctly describe the static deflection of the disc.

From (3.4) and (3.1) we obtain the boundary condition for (3.2): $v_z(t, 0) = {}^1\!/_{96} Q(t)$.

This condition together with (3.3) defines the solution to (3.2):

$$v = -\int_0^{t-z} {}^1\!/_{96}\, Q(s)\, ds - t^2 - z^2 \tag{3.5}$$

Since the relation between w and r has been fixed, conditions (1.2) cannot be satisfied. We will require, then, that they be satisfied in the mean with respect to r. This will result in an equation for Q(t)

$$Q'' + a_2 Q' + 96 a_1 Q = -192 a_2 \tag{3.6}$$

The initial conditions for (3.6) are obtained from (1.3) and they are $Q(0) = Q'(0) = 0$.

Equation (3.6) is the classical equation of damped vibrations; the properties of its solution are well-known and need not be discussed here. The results of an actual solution are plotted in Figs. 2-6 with a dashed line.

The symmetry properties of the problem were hardly used in the derivation of (3.6) and, therefore, the entire reasoning remains valid if the disc is of any sufficiently smooth shape, as long as it occupies the entire bottom of the cylinder — and in that case the coefficients in (3.6) will be different.

4. We will next describe a numerical method for solving the basic problem and will show the results of actual computations. For the special case R=1, a comparison is made with the approximate solution.

The problem is solved numerically using the conventional procedure; i.e., all derivatives are replaced by difference ratios and the resulting system of linear algebraic equations is fed to a computer.

Equation (1.1) with conditions (1.2) and (1.3) is not amenable to a numerical treatment, because the solution is sought in a semiinfinite region and the initial conditions are not continuous, so that the problem must first be reformulated.

The singularity in the initial conditions can be easily extracted. For this purpose, the solution is written as

$$\psi = \psi_1 + \psi_2, \text{ where } \psi_2 = (t - z)^2 / 2 \ \text{ for } \ z \geqslant t,\ \psi_2 = 0 \ \text{ for } \ z < t$$

It is evident from the linearity of the problem that ψ_1 satisfies (1.1) and (1.2) with the initial conditions

$$\psi_1 = -z^2, \quad \psi_{1t} = 0 \quad \text{at} \quad t = 0$$

Moreover, $\psi_1 = \psi$ when $z < t$, specifically when $z = 0$, $t > 0$. Since the disc deflection as a function of time is of primary interest here, we will henceforth make no distinction between ψ and ψ_1.

The initial conditions contain a differential equation in ψ_z and, therefore, it is expedient to change the variables

$$w(t, r) = \psi_z(t, 0, r),$$
$$p(t, z, r) = -\psi_{tt}(t, z, r)$$

This gives an equation for p:

$$p_{tt} = p_{zz} + \Delta p \quad (\Delta p = p_{rr} + p_r / r) \tag{4.1}$$

with the conditions

$$p_r(t, 0, z) = p_r(t, R, z) = 0, \quad p_z(t, r, 0) = -w_{tt}$$
$$p(0, r, z) = 2, \quad p_t(0, r, z) = 0 \tag{4.2}$$

Now $w(t, r) = 0$ for $1 \le r \le R$, and for $r < 1$ the following equation:

$$w_{tt} + a_1 \Delta\Delta w + a_2 p = 0 \tag{4.3}$$

is satisfied by w with the conditions

$$\begin{gathered} w_r(t, 0) = w_{rrr}(t, 0) = 0 \\ w(t, 1) = w_r(t, 1) = 0, \quad w(0, r) = w_t(0, r) = 0 \end{gathered} \tag{4.4}$$

The boundary conditions (4.2) and (4.4) include one for $r = 0$ which is based on symmetry and reduces to the stipulation that p and w must be even functions of r if they do not depend on θ.

Relations (4.1), (4.2), (4.3), and (4.4) are fully equivalent to the original problem for $z < t$.

Inasmuch as a numerical solution is feasible only within a limited region, an additional boundary condition must be entered into the problem. The semiinfinite cylinder is intersected at $z = z_0$, and at this boundary we stipulate that

$$p_t + p_z = 0 \quad \text{for} \quad z = z_0 \tag{4.5}$$

The significance of this stipulation will become clear if we consider the oblique coincidence of a plane wave on a plane surface where (4.5) is satisfied. Let condition (4.5) be stipulated in the xz plane at $z = 0$, and let an incident wave $p = f(t - \cos\alpha z - \sin\alpha x)$ be given where $z < 0$. The reflected wave will then be

$$p = \frac{1 - \cos\alpha}{1 + \cos\alpha} f(t + \cos\alpha z - \sin\alpha x) \tag{4.6}$$

If there is no boundary at $z = 0$, there will be no reflected wave. It is evident from (4.6) that the reflection coefficient becomes zero for normal incidence ($\alpha = 0$) and is approximately α^2 for small angles α.

As is to be expected, at a sufficiently large z the motion will become one-dimensional, and stipulation (4.5) has thus been justified.

The region is bounded with respect to r, but it could happen that $R \gg 1$; in this case we assume $R = 3$, and for $r = 3$ we stipulate a condition analogous to (4.5).

The problem then is to solve Eqs. (4.1)-(4.5) for the region $0 \le r \le R$, $0 \le z \le z_0$, $t > 0$.

At points inside this region Eq. (4.1) is approximated by a triple-layer explicit second-order grid, with the intervals along the time and the space coordinates chosen so as to satisfy the Courant condition [2]. Equation (4.3) is solved by means of an implicit grid. The reason for this is that, with most of the machine time expended on computing p at points inside the region, it is desirable to avoid additional restrictions on the time intervals, which would enter into the computation if an explicit grid were used for (4.3).

We use the following designations: h for intervals along both r and z, and τ for the time intervals,

$$p_{ij}^k = p(k\tau, hi, hj), \quad -2 \leqslant k, \quad 0 \leqslant i \leqslant n_3, \quad -1 \leqslant j \leqslant n_2$$

$$n_3 = R/h, \quad n_2 = z_0/h, \quad w_i^k = w(k\tau, hi), \quad 0 \leqslant i \leqslant n, \quad n = 1/h$$

In order to satisfy the Courant condition, we let $h = 2\tau$.

In order to approximate the m-th derivative of function $f(\xi)$, we introduce central difference operators δ_ξ^m and unilateral difference operators d_ξ^m along ξ, in accordance with

$$\begin{gathered} \delta_\xi^1 f = \frac{1}{2\xi_0}[f(\xi + \xi_0) - f(\xi - \xi_0)], \quad \delta_\xi^2 f = \frac{1}{\xi_0^2}[f(\xi + \xi_0) - 2f(\xi) + f(\xi - \xi_0)] \\ d_\xi^1 f = \frac{1}{2\xi_0}[3f(\xi) - 4f(\xi - \xi_0) + f(\xi - 2\xi_0)] \\ d_\xi^2 f = \frac{1}{\xi_0^2}[2f(\xi) - 5f(\xi - \xi_0) + 4f(\xi - 2\xi_0) - f(\xi - 3\xi_0)] \end{gathered} \tag{4.7}$$

where $\xi_0 = \tau$ if $\xi = t$ and $\xi_0 = h$ if $\xi = r, z$.

For an approximation of Δ we introduce operator $q = \delta_r^2 + r^{-1}\delta_r^1$. It is evident that these operators approximate the respective derivatives within a second-order accuracy.

For Eq. (4.1) we use a triple-layer explicit grid

$$(\delta_t^2 - \delta_z^2 - q)\, p_{ij}^k = 0, \quad 0 \leqslant k, \quad 0 < i < n_3, \quad 0 \leqslant j < n_2 \tag{4.8}$$

The boundary conditions for (4.8) become

$$(d_z^1 + d_t^1) p_{ij}^k = 0 \quad \text{at} \quad j = n_2, \; 0 < i < n_3$$
$$d_{-z}^1 p_{0j}^k = 0 \quad \text{for} \quad 0 \leqslant j < n_2 \tag{4.9}$$
$$(d_r^1 + \alpha d_t^1) p_{ij}^k = 0 \quad \text{at} \quad i = n_3, \; 0 \leqslant j < n_2$$

where $\alpha = 1$ if $R = 3$ and $\alpha = 0$ if $R < 3$.

The boundary condition at $z = 0$ is $p_z + w_{tt} = 0$. For an approximation of this condition we introduce fictitious points $j = -1$ according to the formula

$$p_{i,-1}^k = p_{i,1}^k + 2h d_t^2 w_i^k \tag{4.10}$$

and assume that (4.8) is valid for $j = 0$.

By virtue of the condition $p_z + w_{tt} = 0$, Eq. (4.3) transforms into

$$(1 + a_2 h) w_{tt} + a_1 \Delta\Delta w + a_2 p(t, r, h) + o(h^2) = 0$$

and is approximated by a quadruple-layer implicit grid

$$(1 + a_2 h) d_t^2 w_i^k + a_1 q q w_i^k + a_2 p_{i,1}^k = 0, \; k \geqslant 1 \tag{4.11}$$

The significance of this not quite obvious transformation can be explained on the example of the one-dimensional equation $p_{tt} = p_{zz}$, with the boundary conditions $w_{tt} = -a_2 p$, $p_z + w_{tt} = 0$ at $z = 0$ and the initial conditions $w = w_t = 0$, $p = p_t = 0$ at $t = 0$.

Letting $h = \tau$, we obtain analogously to (4.8)

$$p_j^{k+1} = p_{j+1}^k + p_{j-1}^k - p_j^{k-1}, \quad 0 \leqslant j < \infty, \quad 0 \leqslant k$$
$$p_j^{-1} = p_j^1 = 1, \quad -1 \leqslant j \tag{4.12}$$

The fictitious points $j = -1$ are introduced analogously to (4.10)

$$p_{-1}^k = p_1^k + 2h w_{tt} \tag{4.13}$$

Let us consider two ways of approximating Eq. (4.13). The first way is according to (4.3), the other way is according to (4.11)

$$w_{tt} + a_2 p_0^k = 0, \quad (1 + a_2 h) w_{tt} + a_2 p_1^k = 0 \tag{4.14}$$

We can infer from (4.12) that $p_{j+1}^{k+1} - p_j^k = 0$ for $j \geq 0$, $k \geq 0$. From this we obtain an equation for p_0^k

$$p_0^{k+1} + 2a_2 h p_0^k - p_0^{k-1} = 0 \tag{4.15}$$

based on the first method of approximating the boundary conditions, or

$$p_0^{k+1} - \frac{1 - a_2 h}{1 + a_2 h} p_0^{k-1} = 0 \tag{4.16}$$

based on the second method.

It is evident that (4.16) is stable for any $h \geq 0$ while (4.15) is unstable for every $h > 0$.

The boundary conditions for (4.3) at $r = 1$ are: $w = 0$ and $w_r = 0$. The interval along r has been chosen so as to make $r = 1$ a point on the grid and, therefore, the first of these conditions is satisfied exactly.

We now introduce the fictitious point $r = 1 + h$, and the second boundary condition becomes $w_{n+1} = w_{n-1}$.

In order to approximate the conditions at $r = 0$, we introduce fictitious points $r = -h$ and $r = -2h$ making the boundary conditions $w_{-1} = w_1$ and $w_{-2} = w_2$. Equation (4.11) is assumed valid for $0 \leq i \leq n-1$. Equation (4.3) has a singularity at $r = 0$, however, which makes it necessary to modify (4.11) at $i = 0.1$.

A relatively simple calculation will show that

$$\Delta\Delta w = 16 (w_2 - 4w_1 + 3w_0)/3h^4 + o(h^2) \quad \text{for} \quad r = 0 \tag{4.17}$$
$$\Delta\Delta w = \left(2w_3 - \frac{20}{3} w^2 + \frac{26}{3} w_1 - 4w_0\right)/h^4 + o(h^2) \quad \text{for} \quad r = h$$

Therefore, at $i = 0.1$ we must replace expression qqw_i in (4.11) by the right-hand sides of (4.12).

The initial conditions for $k=0, -1$, and -2 are taken as

$$p_{ij}^{k}=1, \quad 0\leqslant i\leqslant n_2, \quad -1\leqslant j\leqslant n_3, \quad w_i^0=w_i^{-1}=w_i^{-2}=0, \quad 0\leqslant i\leqslant n$$

Equation (4.11) is of the pentadiagonal type and there exists for it an efficient method of iteration [2], which will be used here.

The system of finite-difference equations (4.8)-(4.11) approximates the original system; it is necessary to estimate the incurred error, however, since this error will determine the reliability of the results. Henceforth the function $w_0(t)=w(t, 0)$ will be understood to be the solution.

We consider the static problem, which corresponds to (4.1)-(4.4). This is the problem when $\partial/\partial t\equiv 0$ in (4.1)-(4.4) and the initial conditions are removed. The obvious solution to it is

$$p=2, \quad w(r)=-\frac{a_2}{32a_1}(r^2-1)^2 \tag{4.18}$$

A finite-difference approximation of this problem is obtained by letting $d_t^2=\delta_t^2=0$ in (4.8)-(4.11). The resulting system of equations, too, has a simple solution:

$$p_{ij}=2, \quad w_i=-\frac{a_2}{32a_1}\left(1-\frac{i^2}{n^2}\right)\left(1+\frac{2}{n^2}-\frac{i^2}{n^2}\right) \tag{4.19}$$

From (4.18) and (4.19) we obtain an expression for the approximation error

$$\frac{w_0-w(0)}{w(0)}=\frac{2}{n^2}=2h^2 \tag{4.20}$$

From physical considerations and on the basis of the analysis in section 2, it appears probable that the solution of (4.1)-(4.4) will approach the steady-state solution as $t\to\infty$ and, therefore, (4.20) may be treated as a conservative estimate of the maximum error.

This analysis applies to the error in approximating the space derivatives, and advantage has been taken of the fact that excluding the time dependence simplifies the situation considerably. The time derivatives can be treated in an analogous manner.

Inasmuch as a fitting specific case cannot be found, a model problem was used for evaluation. A one-dimensional problem was considered for Eq. (3.2) with the boundary conditions (3.4) and (3.6).

The problem has an exact solution, which could be obtained from (3.6). Equation (3.2) as well as the boundary and the initial conditions were replaced by finite-difference equations analogous to (4.8)-(4.11). These equations were solved on the interval $0\leq t\leq 10$ and the mean-squared deviation of the resulting solution from the exact solution was calculated.

The numerical value of this deviation for the various parameter values is denoted by ε and shown in Figs. 2-6.

The results obtained for specific values of parameters a_1 and a_2 are shown in Figs. 2-6. Curves 1 represent $w_0(t)$ for $R=1$, $n=n_3=10$, and $n_2=30$, while curves 2 represent $w_0(t)$ for $R=3$, $n=10$, and $n_2=n_3=30$, with the solution to (3.6) shown by a dashed line. Obviously, various values of the dimensional parameters may correspond to the same combination of a_1 and a_2. One may assume that all parameters for the graphs in Figs. 2-4 are the same except the disc thickness, for example, which decreases by a factor of 2.0 from graph to graph, and that the graphs in Figs. 5 and 6 are the same as in Figs. 2 and 3, except for a 10 times higher modulus of elasticity. For sufficiently rigid discs, obviously, all curves will be close and, in particular, curves 1 and 2 in Fig. 2 will merge within plotting accuracy.

In summarizing, we conclude that (3.6) yields a quantitatively satisfactory solution for sufficiently rigid discs and that also otherwise the agreement with the exact solution is qualitatively close.

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